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VI.

ON THE AUTOMORPHIC LINEAR TRANSFORMATION OF A BILINEAR FORM.

By HENRY TABER.

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IN the *Philosophical Transactions* for 1858, Cayley gave the following determination of the general automorphic linear transformation of the alternate bilinear form with cogrediant variables

$$(\Omega\overset{\wedge}{)}(x_1, x_2, \ldots \overset{\wedge}{)}(y_1, y_2, \ldots),$$

namely,

$$(\Omega - \Upsilon)^{-1}(\Omega + \Upsilon),$$

in which Υ denotes an arbitrary symmetric linear transformation.

As shown by Cayley, this is equivalent to the determination of the general solution of the matrical equation $\breve{\phi}\,\Omega\,\phi = \Omega$, in which Ω is a known skew symmetric matrix, and $\breve{\phi}$ denotes the transverse of ϕ. Cayley's solution of this equation fails for those matrices ϕ of which -1 is a latent root.

If Ω is real, and is both skew symmetric and orthogonal (i. e. if $\Omega^{-1} = \breve{\Omega} = -\Omega$), the product of two of Cayley's expressions gives every real solution of this equation.

Thus, let ϕ be any real solution of the equation $\breve{\phi}\,\Omega\,\phi = \Omega$, of which -1 is a latent root. A polynomial $\Phi = f(\phi)$ in integer powers of ϕ with real coefficients can be formed, containing every factor in the identical equation to ϕ, except $\phi + 1$, and such that

$$\Phi^2 = \Phi,\ \breve{\Phi}\,\Omega = \Omega\,\Phi.$$[*]

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[*] The latent roots of ϕ being ± 1 of multiplicity m and n respectively, g_r, g_r^{-1}, each of multiplicity p_r, for $r = 1, 2, \ldots i$, the identical equation to ϕ is

$$(\phi - 1)^m (\phi + 1)^n (\phi - g_1)^{p_1} (\phi - g_1{}^{-1})^{p_1} (\phi - g_2)^{p_2} (\phi - g_2{}^{-1})^{p_2} \ldots = 0;$$

and if

$$F_r(z) = [(z+1)^n - (g_r^{+1})^n]^{pr}\,[(z+1)^n - (g_r^{-1}+1)^n]^{pr},$$

If now

$$\bar{\phi} = \phi\,(1 - 2\,\Phi)\,;$$

then —1 is not a latent root of $\bar{\phi}$, and $\bar{\phi}$ is a solution of the equation $\breve{\phi}\,\Omega\,\phi = \Omega$. The matrix $\phi_0 = (1 - 2\,\Phi)$ is also a solution of this equation; and we have $\phi_0^2 = 1$. Therefore

$$\breve{\phi}_0\,\Omega = \Omega\,\phi_0.$$

But if

$$\Omega\,\phi_0 = \psi$$

we then have

$$\breve{\psi} = -\psi.$$

Since $\phi_0 = \Omega^{-1}\,\psi$, and since $\phi_0^2 = 1$, we have

$$\Omega^{-1}\,\psi\,\Omega^{-1}\,\psi = 1.$$

Therefore,

$$\begin{aligned}\breve{\psi}\,\Omega\,\psi &= -\psi\,\Omega\,\psi\\ &= \psi\,\Omega^{-1}\,\psi\\ &= \Omega.\end{aligned}$$

Consequently ψ is also a solution of the equation.

The matrix $\bar{\phi}$ has no latent root equal to —1. The matrix Φ, and therefore ϕ_0, are real; ψ is then a real skew symmetric matrix, and consequently has no latent root equal to —1. Therefore both $\bar{\phi}$ and ψ may be represented by Cayley's expression. Therefore the most general real solution of the equation $\breve{\phi}\,\Omega\,\phi = \Omega$, in which Ω is a real skew symmetric matrix such that $\Omega^2 = -1$, is

$$\phi = \Omega^{-1}\,(\Omega - \Upsilon)^{-1}\,(\Omega + \Upsilon)\,(\Omega - \Upsilon')^{-1}\,(\Omega + \Upsilon'),$$

in which Υ and Υ' are arbitrary real symmetric matrices. The expression

$$(\Omega - \Upsilon)^{-1}\,(\Omega + \Upsilon)\,(\Omega - \Upsilon')^{-1}\,(\Omega + \Upsilon')$$

is equally general.

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$$\Phi = \frac{[(\phi + 1)^n - (1 + 1)^n]^m}{(-1)^m\,(2)^{mn}}\;\frac{F_1(\phi)}{F_1(-1)}\;\frac{F_2(\phi)}{F_2(-1)}\;\cdots\;\frac{F_i(\phi)}{F_i(-1)}.$$

Since ϕ is real, its imaginary latent roots occur in pairs which are conjugate imaginary, and have the same multiplicity. From which it follows that Φ is real.